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Examples, Problems, and Results in Effect Algebras

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This article discusses various unsolved problems and conjectures that have arisen in the study of effect algebras (or *D*-posets) during the last few years. We also include some examples, counterexamples, and results that motivate or partially solve these problems. The problems mainly concern sharp and principal elements, the existence of infima in Hilbert space effect algebras, tensor products, and interval algebras.

1. INTRODUCTION

Effect algebras (or *D*-posets) have recently been introduced as an algebraic structure for investigating the foundations of quantum mechanics (Cattaneo and Nisticò, 1985; Dvurečenskij, 1995; Dvurečenskij and Pulmannová, 1994; Foulis and Bennett, 1994; Foulis *et al.*, 1994; Giuntini and Greuling, 1989; Greechie and Foulis, 1995; Greechie *et al.*, n.d.; Kôpka, 1992; Kôpka and Chovanec, 1994). This framework gives a unification of the operational (Busch *et al.*, 1991; Davies, 1976; Holevo, 1982; Kraus, 1983; Ludwig, 1983/1985) and quantum logic (Beltrametti and Cassinelli, 1981; Mackey, 1963; Pták and Pulmannová, 1991; Varadarajan, 1968/1970) approaches to quantum mechanics and yields a natural definition of a tensor product, a concept that is necessary for the study of combined physical systems (Dvurečenskij and Pulmannová, 1994; Foulis, 1989). In the last few years, the theory of effect algebras has enjoyed a rapid development. Although many questions have been answered, this period has also produced various unsolved problems and conjectures. Perhaps this is a good time to step back and see what has been done and ask what still needs to be accomplished. This article points out certain gaps in our knowledge. We shall review some of these unsolved

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problems and present various examples, counterexamples, and conjectures. Moreover, we shall include results that motivate and partially solve these problems.

Many investigators in this field can produce a list of significant unsolved problems. It is unavoidable that the present list is biased toward the author's interests and is rather incomplete. In this brief article we concentrate on three main themes. In Section 3 we discuss sharp and principal elements in an effect algebra. Our main concerns will be the relationships between these elements and certain orthoalgebras and orthomodular posets. Section 4 studies Hilbert space effect algebras and the existence of the infimum for two effects. We also consider the existence of a generalized infimum. In Section 5 we inquire into the form of certain tensor products and ask about correspondences between the existence of tensor products and states. Moreover, we discuss relationships between interval algebras and the existence of chain tensor products.

2. BASIC DEFINITIONS

This section briefly reviews some of the basic definitions that will be needed in the sequel. An *effect algebra* is an algebraic system $(P, \oplus, 0, 1)$, where 0 and 1 are distinct elements of P and $\oplus$ is a partial binary operation on P that satisfies the following conditions:

(1) If $a \oplus b$ is defined, then $b \oplus a$ is defined and $b \oplus a = a \oplus b$.
(2) If $a \oplus b$ and $(a \oplus b) \oplus c$ are defined, then $b \oplus c$ and $a \oplus (b \oplus c)$ are defined and $a \oplus (b \oplus c) = (a \oplus b) \oplus c$.
(3) For every $a \in P$ there exists a unique $a' \in P$ such that $a \oplus a' = 1$.
(4) If $a \oplus 1$ is defined, then $a = 0$.

We write $a \leq b$ if there exists a c such that $a \oplus c = b$. It can be shown that $a \oplus b$ is defined if and only if $a \leq b'$. Moreover, $(P, \leq, 0, 1)$ is a bounded poset in which $a'' = a$ and $a \leq b$ implies $b' \leq a'$ (Foulis and Bennett, 1994; Greechie and Foulis, 1995). If $a \leq b'$, we write $a \perp b$. It follows from (2) that we can write $b = a_1 \oplus \cdots \oplus a_n$ without parentheses whenever it is defined. In this case, if $a_i = a$, $i = 1, \ldots, n$, and b is defined, we write $b = na$. An effect algebra P is an orthoalgebra if $a \perp a$ implies that $a = 0$ and P is an orthomodular poset if $a \perp b$ implies that $a \oplus b = a \vee b$. We call P *distributive* if P is lattice ordered and, as a lattice, is distributive.

For effect algebras P, Q a mapping $\phi: P \to Q$ is said to be (1) *additive* if $a \perp b$ implies $\phi(a) \perp \phi(b)$ and $\phi(a \oplus b) = \phi(a) \oplus \phi(b)$; (2) a *morphism* if ϕ is additive and $\phi(1) = 1$; (3) a *monomorphism* if ϕ is a morphism and $\phi(a) \perp \phi(b)$ implies $a \perp b$; (4) an *isomorphism* if ϕ is a surjective monomorphism. Let P, Q, and R be effect algebras. A mapping $\beta: P \times Q \to R$ is a *bimorphism* if for every $a \in P$, $b \in Q$, $\beta(a, \cdot)$ and $\beta(\cdot, b)$ are additive and $\beta(1, 1) = 1$. A bimorphism $\beta: P \times Q \to R$ is *positive* if $\beta(a, b) = 0$ implies that $a = 0$ or $b = 0$ and β is *strong* if $\beta(\cdot, 1)$ and $\beta(1, \cdot)$ are monomorphisms.

A simple example of an effect algebra is $[0, 1] \subseteq \mathbb{R}$, where $a \perp b$ if and only if $a + b \leq 1$, in which case $a \oplus b = a + b$. Another example is an *n-chain*,

$$C_n = \{0, a, 2a, \ldots, na = 1\}$$

A morphism $\phi: P \to [0, 1]$ is called a *state* and we denote the set of states on P by $\Omega(P)$. If $\Omega(P) \neq \phi$, we call P *stately* and if $\Omega(P) = \phi$, we call P *stateless*. There are examples of stateless effect algebras (Greechie, 1971; Gudder and Greechie, n.d.). Also, $[0, 1]$ and C_n are stately and have only one state. A set of states S on P is *positive* if for every $a \in P$, $a \neq 0$, there exists an $s \in S$ such that $s(a) \neq 0$ and S is *order determining* if $s(a) \leq s(b)$ for every $s \in S$ implies $a \leq b$. If $(P, \hat{\oplus}, 0, 1)$ is an effect algebra and $0 \neq u \in P$, let

$$P[0, u] = \{a \in P: 0 \leq a \leq u\}$$

Then $(P[0, u], \oplus, 0, u)$ is an effect algebra, where $a \oplus b$ is defined if and only if $a \hat{\oplus} b \leq u$, in which case $a \oplus b = a \hat{\oplus} b$.

Let P, Q, and T be effect algebras and let $\tau: P \times Q \to T$ be a bimorphism. We call (T, τ) a *tensor product* of P and Q if (1) for every bimorphism $\beta: P \times Q \to R$ there exists a morphism $\phi: T \to R$ such that $\beta = \phi \circ \tau$; (2) every element of T is a finite sum of elements of the form $\tau(a, b)$. The tensor product is unique to within an isomorphism if it exists. We then write $T = P \otimes Q$, $\tau(a, b) = a \otimes b$, and say that $P \otimes Q$ exists. It can be shown that if P and Q admit a bimorphism or if P and Q are stately, then $P \otimes Q$ exists (Dvurečenskij, 1995; Dvurečenskij and Pulmannová, 1994). However, there are effect algebras whose tensor product does not exist (Gudder and Greechie, n.d.). We say that $P \otimes Q$ is *positive* if τ is positive and $P \otimes Q$ is *strong* if τ is strong.

Let G be an additively written, partially ordered Abelian group (Fuchs, 1963; Goodearl, 1986). Let $u \in G$ with $u > 0$ and let

$$P = G^+[0, u] = \{g \in G: 0 \leq g \leq u\}$$

Then P can be organized into an effect algebra $(P, \oplus, 0, u)$ by defining $a \oplus b$ if and only if $a + b \leq u$, in which case $a \oplus b = a + b$. In the effect algebra P we have $a' = u - a$ and the effect algebra partial order on P coincides with the restriction to P of the partial order on G. An effect algebra of the form $G^+[0, u]$ (or isomorphic to an effect algebra of this form) is called an *interval effect algebra* or, for short, an *interval algebra*. Notice that $[0, 1] = R^+[0, 1]$ and $C_n = Z^+[0, n]$ are interval algebras. Since an interval algebra is stately (Bennett and Foulis, n.d.-b), an effect algebra admits a morphism into an interval algebra if and only if it is stately.

The most important example of an effect algebra is a Hilbert space effect algebra. These effect algebras are frequently employed as quantum mechanical models and are useful in quantum measurement theory (Busch *et al.*, 1991; Davies, 1976; Holevo, 1982; Kraus, 1983; Ludwig, 1983/1985). Let H be a complex Hilbert space and let $\mathcal{S}(H)$ be the set of all bounded self-adjoint operators on H. The *positive cone* $\mathcal{S}(H)^+$ in $\mathcal{S}(H)$ is the set of all $A \in \mathcal{S}(H)$ that satisfy $\langle Ax, x\rangle \geq 0$ for all $x \in H$. We then write $A \leq B$ if $B - A \in \mathcal{S}(H)^+$. Letting 0 and 1 be the zero and identity operators, respectively, we have $1 \in \mathcal{S}(H)^+$ and $(\mathcal{S}(H), +, 0, \leq)$ is a partially ordered Abelian group. We call the interval algebra $\mathscr{E}(H) = \mathcal{S}(H)^+[0, 1]$ a *Hilbert space effect algebra*. Notice that if $0 \leq \lambda \leq 1$, $\lambda \in R$, and $A, B \in \mathscr{E}(H)$, then $\lambda A + (1 - \lambda)B \in \mathscr{E}(H)$, so $\mathscr{E}(H)$ is a convex subset of the real vector space $\mathcal{S}(H)$. Any projection is in $\mathscr{E}(H)$ and if M is a closed subspace of H, we denote the projection onto M by P_M.

3. SHARP AND PRINCIPAL ELEMENTS

An element a of an effect algebra P is *sharp* if $a \wedge a' = 0$. Sharp effects correspond to measurements that can be performed with perfect accuracy. It can be shown that an effect algebra is an orthoalgebra if and only if all of its elements are sharp (Foulis and Bennett, 1994). An element $e \in P$ is *principal* if $a, b \leq e$ and $a \perp b$ imply that $a \oplus b \leq e$. If e is principal, it is not hard to show that e is sharp. However, the following example shows that the converse does not hold in general.

Example 3.1 (Wright Triangle). Let

$$P = \{0, 1, a_i, a_i', i = 1, \ldots, 6\}$$

where $\oplus$ is determined by

$$a_1 \oplus a_2 \oplus a_3 = a_3 \oplus a_4 \oplus a_5 = a_5 \oplus a_6 \oplus a_1 = 1$$

Then P is an orthoalgebra, so every element is sharp. However, $a_5' = a_1 \oplus a_6 = a_3 \oplus a_4$ is not principal. Indeed, $a_1, a_3 \leq a_5'$ and $a_1 \perp a_3$, but $a_1 \oplus a_3 = a_2' \not\leq a_5'$.

It can be shown that an effect algebra is an orthomodular poset if and only if every element is principal. Thus, an orthoalgebra that is not an orthomodular poset has a sharp element that is not principal and the Wright triangle is such an example. The following theorem is proved in Bennett and Foulis (n.d.-a).

Theorem 3.1. (a) If an effect algebra P is lattice ordered, then $a \in P$ is sharp if and only if a is principal. (b) The sharp elements of a distributive effect algebra P form a Boolean subalgebra of P.

If for every sharp (principal) $a, b \in P$ with $a \perp b$ we have $a \oplus b$ is sharp (principal), then the sharp (principal) elements of P would form a sub-effect algebra of P and hence an orthoalgebra (orthomodular poset) in P. However, the next example shows that this does not happen in general.

Example 3.2 (Foulis and Greechie). Let $P = \{0, 1, a, b, c, a', b', c'\}$ be the effect algebra with the following $\oplus$ table. In this table we do not include 0 and 1, since they have trivial sums and a dash means that the corresponding $\oplus$ is not defined:

$\oplus$	a	b	c	a'	b'	c'
a	—	c'	b'	1	—	—
b	c'	b'	a'	—	1	—
c	b'	a'	—	—	—	1
a'	1	—	—	—	—	—
b'	—	1	—	—	—	—
c'	—	—	1	—	—	—

We may think of $\oplus$ as being determined by the equations $a \oplus b \oplus c = 1$, $b \oplus b = b'$. Thus, P can be thought of as the Boolean algebra 2^3 with an additional edge so that $b \oplus b = b'$. Now a and c are sharp and principal, but $b' = a \oplus c$ is neither sharp nor principal. Indeed, $b \wedge b' = b$, so b' is not sharp and hence not principal. To see directly that b' is not principal, we have $a, b \leq b'$, but $a \oplus b = c' \nleq b'$. Since $a \vee b$ does not exist, P is not lattice ordered. However, the sharp and principal elements coincide, so the converse of Theorem 3.1(a) does not hold.

We shall see in the next section that the sharp and principal elements in $\mathscr{E}(H)$ coincide and form an orthomodular lattice (and hence an orthoalgebra) in $\mathscr{E}(H)$.

Open Problem 3.1. Characterize the effect algebras whose sharp and principal elements coincide.

Open Problem 3.2. Characterize the effect algebras P whose sharp elements form an orthoalgebra in P (that is, if $a, b \in P$ are sharp and $a \perp b$, then $a \oplus b$ is sharp).

Open Problem 3.3. Characterize the effect algebras P whose principal elements form an orthomodular poset in P (that is, if $a, b \in P$ are principal and $a \perp b$, then $a \oplus b$ is principal).

4. HILBERT SPACE EFFECT ALGEBRAS

Let $\mathscr{E}(H)$ be a Hilbert space effect algebra and let $\mathscr{P}(H) \subseteq \mathscr{E}(H)$ be the set of projections. Kadison (1951) showed that if $E, F \in \mathscr{P}(H)$, then $E \wedge F$ exists and equals P_M, where $M = EH \cap FH$. It follows that $\mathscr{P}(H)$ forms an orthomodular lattice in $\mathscr{E}(H)$. The following theorem is well known.

Theorem 4.1. On $\mathscr{E}(H)$, the following statements are equivalent. (1) A is sharp, (2) A is principal, (3) A is an extreme point, (4) A is a projection.

It follows from Theorem 4.1 that the sharp (and hence principal) elements of $\mathscr{E}(H)$ form an orthomodular lattice in $\mathscr{E}(H)$. For $A, B \in \mathscr{E}(H)$, we denote the projection onto the closure of the range $\overline{AH}$ of A by P_A and the projection onto $\overline{AH} \cap \overline{AH}$ by $P_{A,B}$. If $\dim H \geq 2$, then $\mathscr{E}(H)$ is not a lattice. For example, in $\mathscr{E}(C^2)$, if $A = \text{diag}(1/2, 1/2)$ and $B = \text{diag}(3/4, 1/4)$, then it can be shown that $A \wedge B$ does not exist (Gudder and Greechie, n.d.). The next result is due to Kadison (1951).

Theorem 4.2. For $A, B \in \mathscr{S}(H)$, $A \wedge B$ exists in $\mathscr{S}(H)$ if and only if A and B are comparable (that is, $A \leq B$ or $B \leq A$).

The next problem corresponds to an analogue of Theorem 4.2 for $\mathscr{E}(H)$. In the sequel, we are referring to the infimum as computed within $\mathscr{E}(H)$.

Open Problem 4.1. Characterize the pairs $A, B \in \mathscr{E}(H)$ such that $A \wedge B$ exists.

Since $A \vee B = (A' \wedge B')'$ when either side exists, this would also characterize the pairs $A, B \in \mathscr{E}(H)$ for which $A \vee B$ exists. Various partial results toward solving Open Problem 4.1 have been obtained. The next two theorems solve the problem for the cases $\dim H = 2, 3$ (Gudder and Greechie, n.d.; Gudder and Moreland, n.d.).

Theorem 4.3. Let $A, B \in \mathscr{E}(H)$ with $\dim H = 2$. Then $A \wedge B$ exists if and only if A and B are comparable or either A or B is a multiple of a one-dimensional projection.

Theorem 4.4. Let $A, B \in \mathscr{E}(H)$ with $\dim H = 3$ and let $P = P_{A,B}$. Then $A \wedge B$ exists if and only if one of the following cases holds. (1) A and B are comparable, (2) $\dim P = 1$, (3) $PA = AP$, $PB = BP$, and AP and BP are comparable, (4) there exists a projection P_1 with $\dim P_1 \geq 2$, $P_1A = AP_1$, $P_1B = BP_1$, $\dim P_1BH$, $\dim P_1(A - B)H \leq 1$, and $\dim P_A = 3$.

It can be shown that the four conditions in Theorem 4.4 are mutually independent. The following example illustrates conditions (2)–(4).

Example 4.1. In $\mathscr{E}(\mathbb{C}^3)$ we have

$$\mathrm{diag}(1, 1/2, 0) \wedge \mathrm{diag}(0, 1, 1/2) = \mathrm{diag}(0, 1/2, 0)$$

$$\mathrm{diag}(1, 1/2, 1/3) \wedge \mathrm{diag}(0, 1, 1/2) = \mathrm{diag}(0, 1/2, 1/3)$$

$$\left[\begin{matrix} 1/2 & 0 & 0 \\ 0 & 1/2 & 1/4 \\ 1 & 1/4 & 1/2 \end{matrix}\right) \wedge \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3/8 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 3/8 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1/2 & 1/4 \\ 0 & 1/4 & 1/2 \end{bmatrix} \wedge \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{does not exist}$$

We also have the following partial results that hold for any H (Gudder, n.d.-b). As before, $P = P_{A,B}$.

Theorem 4.5. For $A, B \in \mathscr{E}(H)$, if $\dim P \leq 1$, then $A \wedge B$ exists.

Theorem 4.6. For $A, B \in \mathscr{E}(H)$, suppose that $AP = PA$, $BP = PB$. Then $A \wedge B$ exists if and only if $(AP) \wedge (BP)$ exists. Moreover, in this case $A \wedge B = (AP) \wedge (BP)$. In particular, if AP and BP are comparable, then $A \wedge B$ is the smaller of AP and BP.

The next two theorems give partial results when A, B, and H satisfy certain restrictions (Gudder, n.d.-b; Gudder and Moreland, n.d.).

Theorem 4.7. If H is separable and A and B are commuting effects with pure point spectrum, then $A \wedge B$ exists if and only if AP and BP are comparable. Moreover, in this case $A \wedge B$ is the smaller of AP and BP.

Theorem 4.8. If $\dim H < \infty$ and A and B are invertible, then $A \wedge B$ exists if and only if A and B are comparable.

The last two theorems motivate the following conjectures.

Conjecture 4.1. If A and B are commuting effects, then $A \wedge B$ exists if and only if AP and BP are comparable.

Conjecture 4.2. If A and B are invertible, then $A \wedge B$ exists if and only if A and B are comparable.

For $A \in \mathscr{S}(H)$ we define $|A| = (A^2)^{1/2}$, where $(A^2)^{1/2}$ is the unique positive square root of A^2. For $A, B \in \mathscr{E}(H)$ define the *generalized infimum* $A \sqcap B$ by

$$A \sqcap B = \tfrac{1}{2}(A + B - |A - B|)$$

This generalized infinum has been studied by various investigators (Gudder, n.d.-b; Kadison, 1951; Lahti and Maczynski, 1995; Topping, 1965). Of course, $A \sqcap B$ always exists as an element of $\mathcal{S}(H)$. However, $A \sqcap B$ need not be in $\mathcal{E}(H)$, because $A \sqcap B$ may not be positive. Nevertheless, for important special cases, we do have $A \sqcap B \in \mathcal{E}(H)$. For example, if A and B are comparable or if A and B commute, then $A \sqcap B \in \mathcal{E}(H)$. The following theorem appears in Gudder (n.d.-b) and generalizes a result in Lahti and Maczynski (1995).

Theorem 4.9. For $A, B \in \mathcal{E}(H)$, suppose that $A \sqcap B \in \mathcal{E}(H)$. (a) Then $A \sqcap B$ is a maximal lower bound for A and B in $\mathcal{E}(H)$. (b) If $A \wedge B$ exists, then $A \wedge B = A \sqcap B$.

Open Problem 4.2. Characterize the pairs $A, B \in \mathcal{E}(H)$ such that $A \sqcap B \in \mathcal{E}(H)$.

5. TENSOR PRODUCTS

The mapping τ: $[0, 1] \times [0, 1] \to [0, 1]$ given by $\tau(a, b) = ab$ is the only bimorphism from $[0, 1] \times [0, 1]$ into $[0, 1]$. This also holds for the set of rationals Q in $[0, 1]$. It can be shown that the tensor product of Q with itself is (Q, τ). However, it is embarrassing that we do not know the form of $[0, 1] \otimes [0, 1]$.

Conjecture 5.1. The tensor product of $[0, 1]$ with itself is not $([0, 1], \tau)$.

If this conjecture is true, we are led to the following.

Open Product 5.1. Describe $[0, 1] \otimes [0, 1]$.

More generally, let X be a nonempty set and let $0, 1 \in [0, 1]^X$ be the constant 0 and 1 functions. Then $([0, 1]^X, \oplus, 0, 1)$ becomes an effect algebra when we define $f \oplus g$ if $f(x) + g(x) \leq 1$ for all $x \in X$, in which case $f \oplus g = f + g$. The mapping

$$\tau: \quad [0, 1]^X \times [0, 1]^Y \to [0, 1]^{X \times Y}$$

given by $\tau(f, g)(x, y) = f(x)g(y)$ is a bimorphism. However, $([0, 1]^{X \times Y}, \tau)$ cannot be $[0, 1]^X \otimes [0, 1]^Y$ unless X and Y are finite sets. Even in the finite case, the form of $[0, 1]^X \otimes [0, 1]^Y$ is unknown and the next conjecture is an extension of Conjecture 5.1.

Conjecture 5.2. The tensor product $[0, 1]^X \otimes [0, 1]^Y$ is not $([0, 1]^{X \times Y}, \tau)$.

If this conjecture is true, we are led to the following.

Open Problem 5.2. Describe $[0, 1]^X \otimes [0, 1]^Y$.

Let $H_1 \otimes H_2$ be the usual tensor product of two Hilbert spaces H_1, H_2. For $A \in \mathscr{E}(H)$, $B \in \mathscr{E}(H)$, define $A \otimes B \in \mathscr{E}(H_1 \otimes H_2)$ by $A \otimes B(f_i \otimes g_j) = Af_i \otimes Bg_j$, where f_i, g_j are bases for H_1, H_2, respectively, and extend by linearity and continuity. Then τ: $\mathscr{E}(H_1) \times \mathscr{E}(H_2) \to \mathscr{E}(H_1 \otimes H_2)$ given by $\tau(A, B) = A \otimes B$ is a bimorphism. Again $(\mathscr{E}(H_1 \otimes H_2), \tau)$ cannot be $\mathscr{E}(H_1) \otimes \mathscr{E}(H_2)$ unless dim H_1, dim $H_2 < \infty$ and we are led to the following.

Conjecture 5.3. The tensor product $\mathscr{E}(H_1) \otimes \mathscr{E}(H_2)$ is not $(\mathscr{E}(H_1 \otimes H_2), \tau)$.

Open Problem 5.3. Describe $\mathscr{E}(H_1) \otimes \mathscr{E}(H_2)$.

In universal algebra the following definition of the tensor product of two effect algebras P and Q is natural. If τ: $P \times Q \to T$ is a bimorphism, then (T, τ) is a tensor product of P and Q if for every bimorphism β: $P \times Q \to R$ there exists a unique morphism ϕ: $T \to R$ such that $\beta = \phi \circ \tau$. Let us call this Definition 2 and our original one, Definition 1 of a tensor product. It is not hard to show that Definition 1 implies Definition 2.

Open Problem 5.4. Is a tensor product according to Definition 2 unique up to an isomorphism?

Open Problem 5.5. Does Definition 2 imply Definition 1?

An effect algebra P is a *tensor chain algebra* if $P \otimes C_n$ exists for all $n \notin \mathbb{N}$. The following result is proved in Gudder (n.d.-a).

Theorem 5.1. For an effect algebra P, the following statements are equivalent. (a) P is a tensor chain algebra, (b) $P \otimes C_n$ exists for all $n \in I$, where $I \subseteq \mathbb{N}$ is infinite, (c) P is stately, (d) $P \otimes [0, 1]$ exists, (e) P admits a morphism into an interval algebra.

It follows from Theorem 5.1 that if P is stateless, then $P \otimes [0, 1]$ does not exist and $P \otimes C_n$ does not exist for all but finitely many $n \in \mathbb{N}$. It can be shown that there exist tensor chain algebras that are not interval algebras and it follows that there exist stately effect algebras that are not interval algebras. As mentioned in Section 1, if P and Q are stately, then $P \otimes Q$ exists. However, the following is unknown.

Conjecture 5.4. There exist stateless effect algebras P and Q for which $P \otimes Q$ exists.

In particular, we conjecture that $W_{3,4} \otimes W_{3,4}$ exists, where $W_{3,4}$ is the 3×4 window (Greechie, 1971).

An effect algebra P is a *doubling tensor chain algebra* if $P \otimes C_{2^n}$ exists and is strong for every $n \in \mathbb{N}$. An interval algebra $G^+[0, u]$ is *normal* if $a \in G^+$ satisfies $2^n a \leq 2^n u$ for some $n \notin \mathbb{N}$ implies that $a \leq u$. Most of the common interval algebras are normal. For example, C_n, $[0, 1]$, and $\mathscr{E}(H)$ are normal. In fact, if $\Omega(G^+[0, u])$ is order determining, then $G^+[0, u]$ is normal. However, the diamond $D = \{0, a, b, 1\}$ where $2a = 2b = 1$ and $a \not\perp b$ is an interval algebra (Bennett and Foulis, n.d.-b) that is not normal. The next theorem is proved in Gudder (n.d.-a).

Theorem 5.2. An effect algebra is a doubling tensor chain algebra if and only if it is a normal interval algebra.

Corollary 5.3. If $\Omega(P)$ is order determining, then P is a normal interval algebra.

The converse of Corollary 5.3 does not hold. For example, the nonstandard unit interval $^*[0, 1]$ is a normal interval algebra. However, $\Omega(^*[0, 1])$ contains only one element and this state vanishes on the infinitesimals. This example also shows that an interval algebra need not have a positive set of states.

Open Problem 5.6. Does $\Omega(P)$ positive imply that P is an interval algebra?

An effect algebra P is a *positive tensor chain algebra* if $P \otimes C_n$ exists and is positive for every $n \in \mathbb{N}$.

Open Problem 5.7. Is every interval algebra a positive tensor chain algebra?

Open Problem 5.8. Is every positive tensor chain algebra an interval algebra?

Let P be an effect algebra and suppose there exists a $u \in P$ such that $2^n u = 1$ for some $n \in \mathbb{N}$. If an effect algebra Q is isomorphic to $P[0, u]$, we call P an *n-doubling* of Q. Moreover, we call u an *n-doubling unit* if $2^n a$ exists for $a \in P$ implies that $a \leq u$. If Q is isomorphic to $P[0, u]$, where u is an n-doubling unit, we call P a *strong n-doubling* of Q.

If $Q = G^+[0, u]$ is an interval algebra, then $G^+[0, 2^n u]$ is an n-doubling of Q, so every interval algebra admits an n-doubling for every $n \in \mathbb{N}$. However, the next result shows that an interval algebra need not admit a strong n-doubling for every $n \in \mathbb{N}$ (Gudder, n.d.-a).

Theorem 5.4. An effect algebra Q is a normal interval algebra if and only if Q admits a strong n-doubling for every $n \in \mathbb{N}$.

Open Problem 5.9. If an effect algebra P admits an n-doubling for every $n \in \mathbb{N}$, is P necessarily an interval algebra?

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